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# ON THE INTEGRABILITY OF A DISCRETE ANALOGUE OF THE KAUP–KUPERSHMITZ EQUATION

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**Abstract.** We study a new example of equation obtained as a result of a recent generalized symmetry classification of differential-difference equations defined on five points of one-dimensional lattice. We have established that in the continuous limit this new equation goes into the well-known Kaup–Kupershmidt equation. We have also proved its integrability by constructing an  $L - A$  pair and conservation laws. Moreover, we present a possibly new scheme for deriving conservation laws from  $L - A$  pairs.

**Keywords:** differential-difference equation, integrability, Lax pair, conservation law

**Mathematics Subject Classification:** 37K10, 35G50, 39A10

## 1. INTRODUCTION

We consider the differential-difference equation

$$u_{n,t} = (u_n^2 - 1) \left( u_{n+2} \sqrt{u_{n+1}^2 - 1} - u_{n-2} \sqrt{u_{n-1}^2 - 1} \right), \quad (1)$$

where  $n \in \mathbb{Z}$ , while  $u_n(t)$  is the unknown function of one discrete variable  $n$  and one continuous variable  $t$ , and the index  $t$  denotes time derivative. Equation (1) is obtained as a result of generalized symmetry classification of five-point differential-difference equations

$$u_{n,t} = F(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}), \quad (2)$$

carried out in [8]. Equation (1) coincides with the equation [8, (E17)] up to  $u_n$  and  $t$  scaling.

Equations (2) play an important role in the study of four-point discrete equations on the square lattice, which are very relevant for today, see e.g. [1, 5, 6, 15]. No relation is known between (1) and any other known equation of the form (2). More precisely, we mean relations in the form of the transformations

$$\hat{u}_n = \varphi(u_{n+k}, u_{n+k-1}, \dots, u_{n+m}), \quad k > m, \quad (3)$$

and their compositions, see a detailed discussion of such transformations in [7]. The only information we have at the moment on (1) is that it possesses a nine-point generalized symmetry of the form:

$$u_{n,\theta} = G(u_{n+4}, u_{n+3}, \dots, u_{n-4}).$$

In this article we explore equation (1) in details. We have found in Section 2 its continuous limit, which is the well-known Kaup–Kupershmidt equation [4, 10]:

$$U_\tau = U_{xxxx} + 5UU_{xxx} + \frac{25}{2}U_xU_{xx} + 5U^2U_x, \quad (4)$$

where the indices  $\tau$  and  $x$  denote  $\tau$  and  $x$  partial derivatives. In order to justify the integrability of (1), we construct an  $L - A$  pair in Section 3 and show that it provides an infinity hierarchy of conservation laws in Section 4. In Section 5 we discuss possible generalizations of a construction

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The research is supported by the Russian Science Foundation (project no. 15-11-20007).

scheme for the conservation laws, which has been formulated in Section 4 by example of equation (1).

## 2. CONTINUOUS LIMIT

In a list of equations of the form (2), presented in [8], most of equations go in continuous limit into the Korteweg-de Vries equation. The exceptions are (1) and the following two equations:

$$u_{n,t} = u_n^2(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}) - u_n(u_{n+1} - u_{n-1}), \quad (5)$$

$$u_{n,t} = (u_n + 1) \left( \frac{u_{n+2}u_n(u_{n+1} + 1)^2}{u_{n+1}} - \frac{u_{n-2}u_n(u_{n-1} + 1)^2}{u_{n-1}} + (1 + 2u_n)(u_{n+1} - u_{n-1}) \right), \quad (6)$$

which correspond to equations (E15) and (E16) of [8]. Equation (5) has been known for a long time [17]. Equation (6) has been found recently in [2] and it is related to (5) by a composition of transformations of the form (3). These three equations in the continuous limit correspond to the fifth order equations of the form:

$$U_\tau = U_{xxxxx} + F(U_{xxxx}, U_{xxx}, U_{xx}, U_x, U). \quad (7)$$

There is a complete list of integrable equations of the form (7), see [3, 11, 14]. Two equations play the main role there, namely, (4) and the Sawada-Kotera equation [16]:

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + 5U_xU_{xx} + 5U^2U_x. \quad (8)$$

All the other are transformed into these two by transformations of the form:

$$\hat{U} = \Phi(U, U_x, U_{xx}, \dots, U_{x\dots x}).$$

It has been known [1] that equation (5) goes in the continuous limit into the Sawada-Kotera equation (8). The other results below are new. Using the substitution

$$u_n(t) = \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{16}\varepsilon^2 U \left( \tau - \frac{9}{80}\varepsilon^5 t, x + \frac{2}{3}\varepsilon t \right), \quad x = \varepsilon n, \quad (9)$$

in equation (1), we get at  $\varepsilon \rightarrow 0$  the Kaup-Kupershmidt equation (4).

It is interesting that equation (6) has two different continuous limits. The substitution

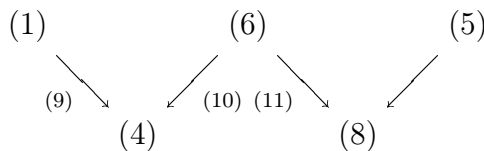
$$u_n(t) = -\frac{4}{3} - \varepsilon^2 U \left( \tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t \right), \quad x = \varepsilon n, \quad (10)$$

in (6) leads to equation (4), while the substitution

$$u_n(t) = -\frac{2}{3} + \varepsilon^2 U \left( \tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t \right), \quad x = \varepsilon n, \quad (11)$$

leads to equation (8). As well as (1), equation (6) deserves further study.

In conclusion, let us present a picture that shows the link between discrete and continuous equations:



3.  $L - A$  PAIR

As the continuous limit shows, equation (1) should be close to equation (5) in its integrability properties. Following the  $L - A$  pair [1, (15,17)], we look for an  $L - A$  pair of the form:

$$L_n \psi_n = 0, \quad \psi_{n,t} = A_n \psi_n \quad (12)$$

with the operator  $L_n$  of the form:

$$L_n = l_n^{(2)} T^2 + l_n^{(1)} T + l_n^{(0)} + l_n^{(-1)} T^{-1},$$

where  $l_n^{(k)}$ ,  $k = -1, 0, 1, 2$ , depend on the finite number of functions  $u_{n+j}$ . Here  $T$  is the shift operator:  $Th_n = h_{n+1}$ . In this case the operator  $A_n$  can be chosen in the form

$$A_n = a_n^{(1)} T + a_n^{(0)} + a_n^{(-1)} T^{-1}.$$

The compatibility condition for the system (12) has the form:

$$\frac{d(L_n \psi_n)}{dt} = (L_{n,t} + L_n A_n) \psi_n = 0 \quad (13)$$

and it must be satisfied on virtue of equations (1) and  $L_n \psi_n = 0$ .

If we suppose that the coefficients  $l_n^{(k)}$  depend on  $u_n$  only, as in [1], then we can see that  $a_n^{(k)}$  depend on  $u_{n-1}, u_n$  only. However, in this case the problem has no solution. Therefore we pass to the case when the functions  $l_n^{(k)}$  depend on  $u_n, u_{n+1}$ . Then the coefficients  $a_n^{(k)}$  must depend on  $u_{n-1}, u_n, u_{n+1}$ . In this case we have managed to find the operators  $L_n$  and  $A_n$  with one irremovable arbitrary constant  $\lambda$ , which plays here the role of spectral parameter:

$$L_n = u_n \sqrt{u_{n+1}^2 - 1} T^2 + u_{n+1} T + \lambda (u_n - u_{n+1} \sqrt{u_n^2 - 1} T^{-1}), \quad (14)$$

$$A_n = \frac{\sqrt{u_n^2 - 1}}{u_n} \left( \sqrt{u_n^2 - 1} (u_{n+1} T + u_{n-1} T^{-1}) - \lambda^{-1} u_{n-1} T + \lambda u_{n+1} T^{-1} \right). \quad (15)$$

The  $L - A$  pair (12,14,15) can be rewritten in the standard matrix form with  $3 \times 3$  matrices  $\tilde{L}_n, \tilde{A}_n$ :

$$\Psi_{n+1} = \tilde{L}_n \Psi_n, \quad \Psi_{n,t} = \tilde{A}_n \Psi_n.$$

Here a new spectral function is given by

$$\Psi_n = 2^{-n} \begin{pmatrix} \frac{\sqrt{u_n^2 - 1}}{u_n} \psi_{n+1} \\ \psi_n \\ \psi_{n-1} \end{pmatrix},$$

and the matrices  $\tilde{L}_n, \tilde{A}_n$  read:

$$\tilde{L}_n = \begin{pmatrix} -\frac{1}{\sqrt{u_n^2 - 1}} & -\frac{\lambda}{u_{n+1}} & \frac{\lambda \sqrt{u_n^2 - 1}}{u_n} \\ \frac{u_n}{\sqrt{u_n^2 - 1}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (16)$$

$$\tilde{A}_n = \begin{pmatrix} \lambda^{-1} - \frac{u_{n-2}}{u_n} \sqrt{u_{n-1}^2 - 1} & u_{n+1} \sqrt{u_n^2 - 1} & \frac{(u_n^2 - 1)(\lambda u_{n+2} \sqrt{u_{n+1}^2 - 1} - u_n)}{u_n^2} \\ u_{n+1} \sqrt{u_n^2 - 1} - \lambda^{-1} u_{n-1} & 0 & \frac{\lambda u_{n+1} \sqrt{u_n^2 - 1} + u_{n-1} (u_n^2 - 1)}{u_n} \\ u_n + \lambda^{-1} u_{n-2} \sqrt{u_{n-1}^2 - 1} & u_n u_{n-1} & \lambda + \frac{u_{n-2}}{u_n} \sqrt{u_{n-1}^2 - 1} \end{pmatrix}. \quad (17)$$

In this case, unlike (13), the compatibility condition can be represented in matrix form:

$$\tilde{L}_{n,t} = \tilde{A}_{n+1} \tilde{L}_n - \tilde{L}_n \tilde{A}_n,$$

without using the spectral function  $\Psi_n$ .

There are two methods to construct the conservation laws by using such matrix  $L - A$  pairs [5, 9, 12]. However, we do not see how to apply those methods in case of the matrices (16) and (17). In the next section, we will use a different scheme for deriving conservation laws from the  $L - A$  pair (12), and that scheme seems to be new.

#### 4. CONSERVATION LAWS

The structure of operators (14,15) allows us to rewrite the  $L - A$  pair (12) in form of the Lax pair. The operator  $L_n$  has the linear dependence on  $\lambda$ :

$$L_n = P_n - \lambda Q_n, \quad (18)$$

where

$$P_n = u_n \sqrt{u_{n+1}^2 - 1} T^2 + u_{n+1} T, \quad Q_n = u_{n+1} \sqrt{u_n^2 - 1} T^{-1} - u_n.$$

Introducing  $\hat{L}_n = Q_n^{-1} P_n$ , we get an equation of the form:

$$\hat{L}_n \psi_n = \lambda \psi_n. \quad (19)$$

The functions  $\lambda \psi_n$  and  $\lambda^{-1} \psi_n$  in the second equation of (12) can be expressed in terms of  $\hat{L}_n$  and  $\psi_n$ , using (19) and its consequence  $\lambda^{-1} \psi_n = \hat{L}_n^{-1} \psi_n$ . As a result we have:

$$\psi_{n,t} = \hat{A}_n \psi_n, \quad (20)$$

where

$$\hat{A}_n = \frac{\sqrt{u_n^2 - 1}}{u_n} \left( \sqrt{u_n^2 - 1} (u_{n+1} T + u_{n-1} T^{-1}) - u_{n-1} T P_n^{-1} Q_n + u_{n+1} T^{-1} Q_n^{-1} P_n \right).$$

It is important that new operators  $\hat{L}_n$  and  $\hat{A}_n$  in the  $L - A$  pair (19,20) do not depend on the spectral parameter  $\lambda$ . For this reason, the compatibility condition can be written in the operator form, without using  $\psi$ -function:

$$\hat{L}_{n,t} = \hat{A}_n \hat{L}_n - \hat{L}_n \hat{A}_n = [\hat{A}_n, \hat{L}_n], \quad (21)$$

i.e. it has now the form of the Lax equation. The difference between this  $L - A$  pair and well-known Lax pairs for the Toda and Volterra equations is that now the operators  $\hat{L}_n$  and  $\hat{A}_n$  are nonlocal. Nevertheless, using the definition of inverse operators, which are linear:

$$P_n P_n^{-1} = P_n^{-1} P_n = 1, \quad Q_n Q_n^{-1} = Q_n^{-1} Q_n = 1, \quad (22)$$

we can check that (21) is true by direct calculation.

The conservation laws of equation (1), which are expressions of the form

$$\rho_{n,t}^{(k)} = (T - 1) \sigma_n^{(k)}, \quad k \geq 0,$$

can be derived from the Lax equation (21), notwithstanding nonlocal structure of the operators  $\hat{L}_n, \hat{A}_n$ , see [18]. For this we must, first of all, represent the operators  $\hat{L}_n, \hat{A}_n$  as formal series in powers of  $T^{-1}$ :

$$H_n = \sum_{k \leq N} h_n^{(k)} T^k. \quad (23)$$

Formal series of this kind can be multiplied according the rule:  $(a_n T^k)(b_n T^j) = a_n b_{n+k} T^{k+j}$ . The inverse series can be obtained by definition (22), for instance:

$$Q_n^{-1} = -(1 + q_n T^{-1} + (q_n T^{-1})^2 + \dots + (q_n T^{-1})^k + \dots) \frac{1}{u_n}, \quad q_n = \frac{u_{n+1}}{u_n} \sqrt{u_n^2 - 1}.$$

The series  $\hat{L}_n$  has the second order:

$$\hat{L}_n = \sum_{k \leq 2} l_n^{(k)} T^k = -(\sqrt{u_{n+1}^2 - 1} T^2 + u_{n+1} u_n T + u_{n+1} u_{n-1} \sqrt{u_n^2 - 1} + \dots).$$

The conserved densities  $\rho_n^{(k)}$  of equation (1) can be found as:

$$\rho_n^{(0)} = \log l_n^{(2)}, \quad \rho_n^{(k)} = \text{res } \hat{L}_n^k, \quad k \geq 1, \quad (24)$$

where the residue of formal series (23) is defined by the rule:  $\text{res } H_n = h_n^{(0)}$ , see [18]. Corresponding functions  $\sigma_n^{(k)}$  can easily be found by direct calculation.

Conserved densities  $\hat{\rho}_n^{(k)}$  below have been found in this way and then simplified in accordance with the rule:

$$\hat{\rho}_n^{(k)} = c_k \rho_n^{(k)} + (T - 1) g_n^{(k)},$$

where  $c_k$  are constant. First three densities of equation (1) read:

$$\begin{aligned} \hat{\rho}_n^{(0)} &= \log(u_n^2 - 1), \\ \hat{\rho}_n^{(1)} &= u_{n+1} u_{n-1} \sqrt{u_n^2 - 1}, \\ \hat{\rho}_n^{(2)} &= (u_n^2 - 1)(2u_{n+2} u_{n-2} \sqrt{u_{n+1}^2 - 1} \sqrt{u_{n-1}^2 - 1} + u_{n+1}^2 u_{n-1}^2) \\ &\quad + u_{n+1} u_{n-1} u_n \sqrt{u_n^2 - 1} (u_{n+2} \sqrt{u_{n+1}^2 - 1} + u_{n-2} \sqrt{u_{n-1}^2 - 1}). \end{aligned}$$

## 5. DISCUSSION OF THE CONSTRUCTION SCHEME

In previous section we have outlined a construction scheme for the conservation laws by example of equation (1). It can easily be generalized to equations of an arbitrarily high order:

$$u_{n,t} = F(u_{n+M}, u_{n+M-1}, \dots, u_{n-M}).$$

Let such equation have an  $L - A$  pair of the form (12) with a linear in  $\lambda$  operator  $L_n$ , and let the operators  $P_n, Q_n$  of (18) have the form:

$$R_n = \sum_{k=k_1}^{k_2} r_n^{(k)} T^k, \quad k_1 \leq k_2 \in \mathbb{Z}, \quad (25)$$

with the coefficients  $r_n^{(k)}$  depending on the finite number of functions  $u_{n+j}$ . We require that

$$\hat{L}_n = Q_n^{-1} P_n = \sum_{k \leq N} l_n^{(k)} T^k$$

has a positive order  $N \geq 1$ . If  $N \leq -1$ , then we change  $\lambda \rightarrow \lambda^{-1}$  and introduce  $\tilde{L}_n = P_n^{-1} Q_n$  of a positive order. In case  $N = 0$  the scheme does not work.

As  $\lambda^k \psi_n = \hat{L}_n^k \psi_n$  for any integer  $k$ , we can consider operators  $A_n$  of the form:

$$A_n = \sum_{k=m_1}^{m_2} a_n^{(k)} [T] \lambda^k, \quad m_1 \leq m_2 \in \mathbb{Z},$$

where  $a_n^{(k)} [T]$  are operators of the form (25). Then we can rewrite  $A_n$  as:

$$\hat{A}_n = \sum_{k=m_1}^{m_2} a_n^{(k)} [T] L_n^k = \sum_{k \leq \hat{N}} \hat{a}_n^{(k)} T^k.$$

We are led to the Lax equation (21) with  $\hat{L}_n, \hat{A}_n$  of the form (23) and, therefore, we can construct the conserved densities as written above, namely, in accordance with (24) with the only difference:  $\rho_n^{(0)} = \log l_n^{(N)}$ .

It should be remarked that the scheme can easily be applied to equation (5) with the  $L - A$  pair [1, (15,17)].

This scheme can also be applied in a quite similar way in the continuous case, namely, to PDEs of the form

$$u_t = F(u, u_x, u_{xx}, \dots, u_{x\dots x}).$$

We consider the operators (25) with  $D_x$  instead of  $T$ , which become the differential operators, where  $D_x$  is the operator of total  $x$ -derivative. Besides,  $k_2 \geq k_1 \geq 0$  and the coefficients  $r_n^{(k)}$  depend on a finite number of functions  $u, u_x, u_{xx}, \dots$ . Instead of (23) we consider the formal series in powers of  $D_x^{-1}$ . A theory of such formal series and, in particular, a definition of the residue are discussed in [13].

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